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RESEARCH REPORT

A General Algorithm For Pade Approximation To Formal Power Series

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A GENERAL ALGORITHM FOR PADE APPROXIMATION TO FORMAL POWER SERIES.

1 INTRODUCTION

A general algorithm for constructing an interpolating function from a linear family of functions forming a Chebyshev system has been given by Brezinski in [1]. This algorithm was extended to generalized rational interpolation [7] and is applied to the Pade-type approximants for the series of functions in [1]. A question arises as to whether this algorithm could be extended to generalize the Pade approximation to a formal power series $f(x)$. Before the discussion, we recall some basic definitions.

The Pade approximant $(m,n;x)$ to a function $f(x)$ which can be expressed as a formal power series $\sum_{k=0}^{\infty} C_k x^k$ is the ratio of $Q(m,n;x)/P(m,n;x)$, where $(m,n;x)$ agrees with the power series as far as possible. The denominator $P(m,n;x)$ and the numerator $Q(m,n;x)$ are polynomials of degree m and n respectively. These polynomials $P(m,n;x)$ and $Q(m,n;x)$ are determined from

$$P(m,n;x) \sum_{k=0}^{\infty} C_k x^k - Q(m,n;x) = 0(x^{m+n+1})$$

by neglecting higher order terms. They can be expressed respectively as the quotient of a determinant of order m+1, and its minor whose elements are formed from the C_i,

$$Q(m,n;x) = \frac{\begin{vmatrix} x^m P_0^{n-m}(x) & x^{m-1} P_0^{n-m+1}(x) & \cdots & P_0^n(x) \\ C_{n-m+1} & C_{n-m+2} & \cdots & C_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ C_n & C_{n+1} & \cdots & C_{n+m} \end{vmatrix}}{D_{m,n}} \qquad (1.1)$$

where $P_0^i(x) = \sum_{k=0}^{i} C_k x^k$, $C_i = 0$ if $i < 0$ and $D_{m,n}$ is the minor obtained by eliminating the first row and the last column of the numerator.

$P(m,n;x)$ is obtained by replacing $P_0^i(x)$ in (1.1) by $1 \ \forall\ i$.

The Pade table of $f(x)$ is a doubly indexed array of the rational function $(m,n;x)$. The array is following

$$\begin{array}{llll} (0,0) & (1,0) & (2,0) & \ldots\ldots (m,0) \\ (0,1) & (1,1) & (2,1) & \ldots\ldots\ldots\ldots \\ (0,2) & (1,2) & (2,2) & \ldots\ldots\ldots\ldots \\ \vdots & \vdots & \vdots & \\ (0,n) & & & \end{array}$$

The partial sums of the power series $\sum_{k=0}^{\infty} C_k x^k$ occupy the first column of the table.

A Pade table is normal if none of the $(m,n;x)$ is nil or equal to each other. In other words, the power series $f(x)$ is normal if, for each pair (m,n), $(m,n;x)$ agrees exactly through the power x^{m+n} and none of the $D_{m,n} = 0$. In particular each coefficient C_i must not vanish. The power series $\sum_{k=0}^{\infty} C_k x^k$ is semi normal if $D_{m,n} = 0$ for $m+n$ even. The nil elements of $(m,n;x)$, if they occur, are isolated by the non-nil elements. Otherwise, the series is called non-normal. More detailed definitions, properties and theorems are given in [6].

Different algorithms for constructing the elements of the Pade table of a power series for the normal case are found in [4]. Some of these algorithms have been modified, in particular situations for the non-normal case [2], [3], [5].

In this paper, the algorithm for interpolation given by Brezinski in [1] is extended to compute the elements of the Pade table of a power series in the normal case. This algorithm is used to compute separately, column by column, the denominator $P(m,n;x)$ and the numerator $Q(m,n;x)$ of the Pade approximant of $f(x)$. In the next section, how this algorithm can be applied to (1.1) will be shown. This algorithm is extended to the seminormal case in Section 3. This extension is similar to the method used to solve the singular case in [7]. In the last section some comments are made about the non-normal case.

2 THE ALGORITHM

The following algorithm is based on the MNA algorithm in [1]. It is convenient to change some of the notation and initialization. $P_m^n(x)$ is used to compute $P(m,n;x)$ and $Q(m,n;x)$ separately by the same algorithm. Similarly $g_{m,i}^n(x)$ is used to compute $Pg_{m,i}^n(x)$ and

$Qg^{n}_{m,i}(x)$ which are used for computing $P(m,n;x)$ and $Q(m,n;x)$ respectively.

Step 1. Initialization

For $j = 0, 1 \ldots n$

$$P^{j}_{1}(x) = \frac{\begin{vmatrix} xP^{j-1}_{0}(x) & P^{j}_{0}(x) \\ C_j & C_{j+1} \end{vmatrix}}{C_j} \qquad (2.1)$$

For $i = 2,3 \ldots m$

$$g^{j}_{1,i}(x) = \frac{\begin{vmatrix} xP^{j-1}_{0} & x^{i}P^{j-i}_{0}(x) \\ C_j & C_{j-i+1} \end{vmatrix}}{C_j}$$

where $P^{j}_{0}(x) = \sum_{k=0}^{j} C_k x^k, \quad j \geqslant 0$

$= \quad 0 \qquad j < 0$

for computing the numerator $P(m,n;x)$

and $P^{j}_{0}(x) = 1, \ \forall\, j$

for computing the denominator $Q(m,n;x)$

Step 2. For $k = 2, 3 \ldots m;$

$j = 0, 1 \ldots n-1$

Compute $$P^{j}_{k}(x) = \frac{\Delta P^{j}_{k-1}(x)}{\Delta g^{j}_{k-1,k}(x)} g^{j}_{k-1,k}(x) - P^{j}_{k-1}(x) \qquad (2.2)$$

If $k = m$ stop otherwise

$$g^{j}_{k,i}(x) = \frac{\Delta g^{j}_{k-1,i}(x)}{\Delta g^{j}_{k-1,k}(x)} g^{j}_{k-1,k}(x) - g^{j}_{k-1,i}(x)$$

$$i = k+1, \ldots m \qquad (2.3)$$

where Δ represents the forward difference on the index j.

$P_k^j(x)$ has the form (1.1) if k,j is replaced by m,n

and

$$g_{k,i}^j(x) = \frac{\begin{vmatrix} x^k P_0^{j-k}(x) & x^{k-1} P_0^{j-k+1}(x) & \cdots & x P_0^{j-1}(x) & x^i P_0^{j-i}(x) \\ c_{j-k+1} & c_{j-k+2} & \cdots & c_j & c_{j-i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{j-1} & c_j & \cdots\cdots\cdots & c^{j+k-2} & c_{j+k-i-1} \end{vmatrix}}{D_{k,j}}$$

The way of computing $g_{k,i}^j(x)$ in (2.2) can be generalized as follows. Instead of using $g_{k,i}^j(x)$ for k = 2,3 ...m, we initialize

$$Hg_{2-k,k}^j(x) = -\frac{\begin{vmatrix} x^k P_0^{j-k} & x^{k-1} P_0^{j-k+1} \\ c_{j-k+1} & c_{j-k+2} \end{vmatrix}}{c_{j-k+2}} \quad j = k-2,\ k-1, \ldots n.$$

$$Vg_{2-k,k}^j(x) = \frac{\begin{vmatrix} x^k P_0^{j-k} & x^{k-1} P_0^{j-k+1} \\ c_{j-k+1} & c_{j-k+2} \end{vmatrix}}{c_{j-k+1}} \quad j = k-1, k \ldots n$$

For $\ell = 3,4 \ldots$

$i = 0,1,2 \ldots$

$k = i+\ell \ldots m$

$Hg_{-i,k}^{i}(x) = -x^k P^{i-k}(x)$.

For j = i+1,...n

Compute $Hg^{j}_{-i,k}(x) = \frac{\Delta Hg^{j}_{-(i+1),k}(x)}{\Delta Vg^{j}_{-i,k-1}(x)} Vg^{j}_{-i,k-1}(x)$

$$- Hg^{j}_{-(i+1),k}(x) \qquad (2.4)$$

$$Vg^{j}_{-i,k}(x) = \frac{\Delta Vg^{j}_{-i,k-1}(x)}{\Delta Hg^{j}_{-(i+1),k}(x)} Hg^{j}_{-(i+1),k}(x)$$

$$- Vg^{j}_{-i,k-1}(x) \qquad (2.5)$$

For $k = 2,3 \ldots m$

$j = 0,1 \ldots n-1$

Compute $P^{j}_{k}(x) = \frac{\Delta P^{j}_{k-1}(x)}{\Delta Hg^{j}_{0,k}(x)} Hg^{j}_{0,k}(x) - P^{j}_{k-1}(x) \qquad (2.6)$

Note: $Vg^{j}_{-i,k}(x) = 0$ and $Hg^{j}_{-i,k}(x) = 0$ when $i \geqslant k$

$Hg^{j}_{0,k}(x) = g^{j}_{k-1,k}(x)$.

The ratios of the differences in (2.2) - (2.6) can easily be shown to be constants by using Sylvester's identity (see [7]). Thus the computation is straightforward. By using this algorithm the Pade table can be built up column by column.

Example: Pade approximants to e^x.

n	${}^Q_P g^n_{1,2}(x)\ (Hg^n_{0,2}(x))$	${}^Q_P g^n_{2,3}(x)\ (Hg^n_{0,3}(x))$	${}^Q_P g^n_{3,4}(x)\ (Hg^n_{0,4}(x))$
0	$\dfrac{0}{x^2}$	$\dfrac{0}{x^3}$	$\dfrac{0}{x^4}$
1	$\dfrac{-x}{-x+x^2}$	$\dfrac{2x}{2x-2x^2+x^3}$	$\dfrac{-6x}{-6x+6x^2-3x^3+x^4}$
2	$\dfrac{-2x-x^2}{-2x+x^2}$	$\dfrac{6x+2x^2}{6x-4x^2+x^3}$	$\dfrac{-24x-6x^2}{-24x+18x^2-6x^3+x^4}$
3	$\dfrac{-3x+2x^2-\frac{1}{2}x^3}{-3x+x^2}$	$\dfrac{12x+6x^2+x^3}{12x-6x^2+x^3}$	$\dfrac{-60x-24x^2-3x^3}{-60x-36x^2-9x^3+x^4}$
4	$\dfrac{-4x-3x^2-x^3-\frac{1}{6}x^4}{-4x+\frac{1}{24}x^2}$	$\dfrac{20x+12x^2+3x^3+\frac{1}{3}x^4}{20x-8x^2+x^3}$	$\vdots$
5	$\dfrac{-5x-4x^2-\frac{3}{2}^3-\frac{1}{3}x^4-\frac{1}{24}x^5}{-5x+x^2}$	$\vdots$	
	$\vdots$		

n	$P^n_1(x)$ (ie. $\frac{Q(1,n;x)}{P(1,n;x)}$)	$P^n_2(x)$	$P^n_3(x)$	$P^n_4(x)$
0	$\dfrac{1}{1-x}$	$\dfrac{1}{1-x+\frac{1}{2}x^2}$	$\dfrac{1}{1-x+\frac{1}{2}x^2-\frac{1}{6}x^3}$	$\dfrac{1}{1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4}$
1	$\dfrac{1+\frac{1}{2}x}{1-\frac{1}{2}x}$	$\dfrac{1+\frac{1}{3}x}{1-\frac{2}{3}x+\frac{1}{6}x^2}$	$\dfrac{1+\frac{1}{4}x}{1-\frac{3}{4}x+\frac{1}{4}x^2-\frac{1}{24}x^3}$	$\dfrac{1+\frac{1}{5}x}{1-\frac{4}{5}x+\frac{3}{10}x^2-\frac{1}{15}x^3+\frac{1}{120}x^4}$
2	$\dfrac{1+\frac{2}{3}x+\frac{1}{6}x^2}{1-\frac{1}{3}x}$	$\dfrac{1+\frac{1}{2}x+\frac{1}{12}x^2}{1-\frac{1}{2}x+\frac{1}{12}x^2}$	$\dfrac{1+\frac{2}{5}x+\frac{1}{20}x^2}{1-\frac{3}{5}x+\frac{3}{20}x^2-\frac{1}{60}x^3}$	$\dfrac{1+\frac{1}{3}x+\frac{1}{30}x^2}{1-\frac{2}{3}x+\frac{1}{5}x^2-\frac{1}{30}x^3+\frac{1}{360}x^4}$
3	$\dfrac{1+\frac{3}{4}x+\frac{1}{4}x^2+\frac{1}{24}x^3}{1-\frac{1}{4}x}$	$\dfrac{1+\frac{3}{5}x+\frac{3}{20}x^2+\frac{1}{60}x^3}{1-\frac{2}{5}x+\frac{1}{20}x^2}$	$\dfrac{1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{120}x^3}{1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3}$	$\vdots$
4	$\dfrac{1+\frac{4}{5}x+\frac{3}{10}x^2+\frac{1}{15}x^3+\frac{1}{120}x^4}{1-\frac{1}{5}x}$	$\dfrac{1+\frac{2}{3}x+\frac{1}{5}x^2+\frac{1}{30}x^3+\frac{1}{360}x^4}{1-\frac{1}{3}x+\frac{1}{30}x^2}$	$\vdots$	$\vdots$
5	$\dfrac{1+\frac{5}{6}x+\frac{1}{3}x^2+\frac{1}{12}x^3+\frac{1}{72}x^4+\frac{1}{720}x^5}{1-\frac{1}{6}x}$	$\vdots$		

3 SEMI-NORMAL CASE

The above algorithm may break down if $C_i = 0$ in (2.1) or $\Delta g^{j}_{k-1,k}(x) = 0$ in (2.2-2.3) or $\Delta Hg^{j}_{-i,k}(x)$ (or $\Delta Vg^{j}_{-i,k}(x)) = 0$ in (2.4-2.6). For the semi-normal case, the extension of (2.4-2.6) allows the recursive algorithm to carry on.

If $\Delta Vg^{j}_{-i,k-1}(x) = 0$ and $\Delta Hg^{j}_{-i,k-1}(x) = 0$ (refer to corollary 4 in [7])) in (2.4), $Hg^{j}_{-i,k}(x)$, $Vg^{j}_{-(i-1),k-1}(x)$ and $P^{j}_{k-1}(x)$ cannot be computed. We have to jump a step to compute $Hg^{j-1}_{-(i-1),k}(x)$, $Hg^{j}_{-(i-1),k}(x)$, $Vg^{j-1}_{-(i-1),k}(x)$, $Vg^{j}_{-(i-1),k}(x)$, $P^{j-1}_{k}(x)$ and $P^{j}_{k}(x)$ by the following

a) $$Hg^{j-1}_{-(i-1),k}(x) = \Delta' g \cdot Vg^{j-1}_{-(i-1),k-1}(x) - Hg^{j-1}_{-i,k}(x)$$

$$Vg^{j-1}_{-(i-1),k}(x) = \frac{1}{\Delta' g} Hg^{j-1}_{-i,k}(x) - Vg^{j-1}_{-(i-1),k-1}(x)$$

$$Hg^{j}_{-(i-1),k}(x) = \Delta g \; Vg^{j+1}_{-(i-1),k-1}(x) - Hg^{j+1}_{-i,k}(x)$$

$$Vg^{j}_{-(i-1),k}(x) = \frac{1}{\Delta g} Hg^{j+1}_{-i,k}(x) - Vg^{j+1}_{-(i-1),k-1}(x) \quad i = 1,2\ldots$$

where $\Delta g = \Delta' g = \dfrac{C_{j-k+1}}{C_{j-k+3}}$ for $k = i+3$.

$$\Delta g = -\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \quad \text{for } k = i+4,\ i+5,\ \ldots$$

where $$\Delta_1 = \frac{Vg^{j+1}_{-1,k-2}(x)}{\Delta Vg^{j}_{-(i-1),k-2}(x)}$$

$$\Delta_2 = \frac{\Delta Hg^{j}_{-(i+1),k}(x)}{Hg^{j+1}_{-(i+1),k-1}(x)} \cdot \frac{\Delta Hg^{j}_{-(i+1),k-1}(x)}{\Delta Vg^{j}_{-i,k-2}(x)} \cdot \frac{\Delta Hg^{j+1}_{-(i+1),k-1}(x)}{\Delta Vg^{j+1}_{-i,k-2}(x)}$$

$$\Delta_3 = \frac{\Delta Hg^{j+2}_{-(i+1),k-1}(x)}{\Delta Vg^{j+2}_{-i,k-2}(x)} \cdot \frac{\Delta Vg^{j+1}_{-i,k-1}(x)}{Hg^{j+2}_{-(i+1)k-1}(x)} \cdot \frac{Vg^{j+2}_{-i,k-2}(x)}{\Delta Hg^{j+1}_{-i,k-1}(x)}$$

and by the same method, we have

$$\Delta' g = -\Delta_1 \cdot \Delta_2 \cdot \Delta'_3$$

where $$\Delta'_3 = \frac{\Delta Hg^{j-1}_{-(i+1),k-1}(x)}{\Delta Vg^{j-1}_{-i,k-2}(x)} \cdot \frac{\Delta Vg^{j-1}_{-i,k-1}(x)}{Hg^{j}_{-(i+1),k-1}(x)} \cdot \frac{Vg^{j}_{-i,k-2}(x)}{\Delta Hg^{j-1}_{-i,k-1}(x)}$$

b) $P^{j-1}_{k}(x) - \Delta_P{}' \; Hg^{j-1}_{0,k}(x) - P^{j-1}_{k-1}(x)$

$P^{j}_{k}(x) = \Delta_P \; Hg^{j+1}_{0,k}(x) - P^{j+1}_{k-1}(x)$

where $\Delta_P = \Delta'_P = \dfrac{C_{j+1}}{C_{j-1}}$ for $k = 2$.

$$\Delta_P = -\Delta'_1 \frac{1}{\Delta_2 . \Delta_3}$$

and $$\Delta'_P = -\Delta'_1 \frac{1}{\Delta_2 . \Delta'_3}$$

where $\Delta'_1 = \dfrac{\Delta P^{j}_{k-2}(x)}{Vg^{j+1}_{0,k-2}(x)}$ for $k = 3,4 \ldots$

A more detailed theoretical derivation is given in [7]. The idea of this extension is similar to Theorem 7.4 of [6]

4 NON-NORMAL CASE

For the non-normal case, the nil elements appear in the Pade table more than once. Hence it will be more complicated to generalize. An algorithm and some techniques for computing the non-normal staircase Pade table can be found in [3], [5]. For the more general scheme, perhaps we can apply the general Neville-Aitken algorithm in [1]. That is

$$P^{j}_{k+m}(x) = \frac{\begin{vmatrix} P^{j}_{m}(x) & \cdots & P^{j+k}_{m}(x) \\ g^{j}_{m,m+1}(x) & \cdots & g^{j+k}_{m,m+1}(x) \\ \cdots & \cdots & \cdots \\ g^{j}_{m,m+k}(x) & \cdots & g^{j+k}_{m,m+k}(x) \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & \cdots \\ g^{j}_{m,m+1}(x) & & g^{j+k}_{m,m+1}(x) \\ \cdots & \cdots & \\ g^{j}_{m,m+k}(x) & \cdots & g^{j+k}_{m,m+k}(x) \end{vmatrix}} = \frac{\begin{vmatrix} P^{j}_{k}(x) & \cdots & P^{j+m}_{k}(x) \\ g^{j}_{k,k+1}(x) & \cdots & g^{j+m}_{k,k+1}(x) \\ \cdots & \cdots & \cdots \\ g^{j}_{k,k+m}(x) & \cdots & g^{j+m}_{k,k+m}(x) \end{vmatrix}}{\begin{vmatrix} 1 & \cdots \quad \cdots & 1 \\ g^{j}_{k,k+1}(x) & & g^{j+m}_{k,k+1}(x) \\ \cdots & \cdots & \\ g^{j}_{k,k+m}(x) & \cdots & g^{j+m}_{k,k+m}(x) \end{vmatrix}}$$

This relation holds for $g^{j}_{k+m,i}(x)$ if we replace the first row in the numerators by $(g^{j}_{m,i}(x) \ \ldots \ g^{j+k}_{m,i}(x))$ and $(g^{j}_{k,i}(x) \ \ldots \ g^{j+m}_{k,i}(x))$ respectively where $k,m = 1,2\ldots$ It can be used in the semi-normal case when $m = 2$ in the second relation, but for $m > 2$, we have to evaluate a larger determinant. The simplification of the computation for more general non-normal cases is still being studied.

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